CAMBRIDGE INTERNATIONAL EXAMINATIONS

GCE Advanced Subsidiary Level and GCE Advanced Level

MARK SCHEME for the May/June 2013 series

9231 FURTHER MATHEMATICS

9231/23 Paper 2, maximum raw mark 100

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

Cambridge will not enter into discussions about these mark schemes.

Cambridge is publishing the mark schemes for the May/June 2013 series for most IGCSE, GCE Advanced Level and Advanced Subsidiary Level components and some Ordinary Level components.

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Mark Scheme Notes

Marks are of the following three types:

M Method mark, awarded for a valid method applied to the problem. Method marks are not lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, e.g. by substituting the relevant quantities into the formula. Correct application of a formula without the formula being quoted obviously earns the M mark and in some cases an M mark can be implied from a correct answer.

A Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated method mark is earned (or implied).

B Mark for a correct result or statement independent of method marks.

- When a part of a question has two or more "method" steps, the M marks are generally independent unless the scheme specifically says otherwise; and similarly when there are several B marks allocated. The notation DM or DB (or dep*) is used to indicate that a particular M or B mark is dependent on an earlier M or B (asterisked) mark in the scheme. When two or more steps are run together by the candidate, the earlier marks are implied and full credit is given.

- The symbol √ implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A or B marks are given for correct work only. A and B marks are not given for fortuitously "correct" answers or results obtained from incorrect working.

- Note: B2 or A2 means that the candidate can earn 2 or 0.
  B2/1/0 means that the candidate can earn anything from 0 to 2.

  The marks indicated in the scheme may not be subdivided. If there is genuine doubt whether a candidate has earned a mark, allow the candidate the benefit of the doubt. Unless otherwise indicated, marks once gained cannot subsequently be lost, e.g. wrong working following a correct form of answer is ignored.

- Wrong or missing units in an answer should not lead to the loss of a mark unless the scheme specifically indicates otherwise.

- For a numerical answer, allow the A or B mark if a value is obtained which is correct to 3 s.f., or which would be correct to 3 s.f. if rounded (1 d.p. in the case of an angle). As stated above, an A or B mark is not given if a correct numerical answer arises fortuitously from incorrect working. For Mechanics questions, allow A or B marks for correct answers which arise from taking *g* equal to 9.8 or 9.81 instead of 10.

The following abbreviations may be used in a mark scheme or used on the scripts:

AEF Any Equivalent Form (of answer is equally acceptable)

AG Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid)

BOD Benefit of Doubt (allowed when the validity of a solution may not be absolutely clear)

CAO Correct Answer Only (emphasising that no "follow through" from a previous error is allowed)

CWO Correct Working Only – often written by a “fortuitous” answer

ISW Ignore Subsequent Working

MR Misread

PA Premature Approximation (resulting in basically correct work that is insufficiently accurate)

SOS See Other Solution (the candidate makes a better attempt at the same question)

SR Special Ruling (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance)

Penalties

MR –1 A penalty of MR –1 is deducted from A or B marks when the data of a question or part question are genuinely misread and the object and difficulty of the question remain unaltered. In this case all A and B marks then become "follow through √" marks. MR is not applied when the candidate misreads his own figures – this is regarded as an error in accuracy. An MR–2 penalty may be applied in particular cases if agreed at the coordination meeting.

PA –1 This is deducted from A or B marks in the case of premature approximation. The PA –1 penalty is usually discussed at the meeting.

Question Number	Mark Scheme Details			Part Mark	Total
1	*EITHER*: Use $I = Ft$ to find impulse I:	$I = 1500 \times 0.01$ [= 15]	M1 A1		
	Use $I = m(v_1 - v_2)$ to find mass m:	$m = I/250 = 0.06$	M1 A1		
	OR: Use $v_2 = v_1 - at$ to find deceleration a:	$a = 250/0{\cdot}01$ [=250000]	(M1 A1)		
	Use $F = ma$ to find mass m:	$m = 1500/a = 0.06$	(M1 A1)		
	OR: Use $s = ½ (v_1 + v_2)t$ to find distance s:	$s = ½ \times 310 \times 0{\cdot}01$ [= 1·55]	(M1 A1)		
	Use $Fs = ½\, m(v_1^2 - v_2^2)$ to find mass m:	$m = 2 \times 1500s/77500 = 0{\cdot}06$	(M1 A1)	4	**4**
	S.R Taking $v_1 - v_2 = 280 + 30$ in any method:	$m = I/310 = 0{\cdot}048[3]$ (max 2/4)	(M1 A1)		
2	State or imply reaction R is zero when contact lost		M1		
	Use $F = ma$ radially when contact lost:	$mv^2/a = mg \cos \theta$ [$- R$]	M1		
	Use $\cos \theta = 5/6$ to find v^2:	$v^2 = ag \cos \theta = 5ag/6$	A1		
	Use conservation of energy at θ: If found by $v^2 = u^2 + 2gh$ lose this A1 only)	$½mv^2 - ½mu^2$ $= mg\{a/8 + a(1 - \cos \theta)\}$	M1 A1		
		$= 7mag/24$ (A.E.F.)	A1		
	Combine to find u:	$u^2 = 5ag/6 - 2 \times 7ag/24$	M1		
		$u = \sqrt{(¼\, ag)}$ *or* $½ \sqrt{(ag)}$	A1	8	**8**
3	Use conservation of momentum, e.g.:	$mv_A + 2mv_B = mu$	B1		
	Use restitution (must be consistent with prev. eqn.):	$v_A - v_B = - eu$	B1		
	Find speed of B after striking barrier (ignore sign):	$v_B' = ½\, v_B$	M1		
	Relate K.E. before and after collision:	$(½mu^2)/9 = ½mv_A^2 + ½(2m)v_B'^2$	M1		
	EITHER: Solve first two eqns for v_A and v_B (A.E.F):	$v_A = ⅓(1 - 2e)u,\ v_B = ⅓(1 + e)u$	M1 A1		
	Substitute for v_A, v_B' in KE eqn:	$u^2/9 = (1 - 2e)^2u^2/9 + ½(1 + e)^2 u^2/9$	A1		
	Simplify and solve for e:	$9e^2 - 6e + 1 = 0,\ e = ⅓$	M1 A1		
	OR: Use $v_A + 2v_B = u$ in KE eqn to give e.g.:	$81v_A^2 - 18uv_A + u^2 = 0$ *or* $81v_B^2 - 72uv_B + 16u^2 = 0$	(M1 A1)		
	Solve for v_A and v_B:	$v_A = u/9$ *and* $v_B = 4u/9$	(A1)		
	Find e from restitution eqn:	$e = (4u/9 - u/9)/u = ⅓$	(M1 A1)	9	**9**

4	Find eqn. of motion at general point: (wrong sign loses this A1 only)	$m\,d^2x/dt^2 = mg(2a - x)/2a - mg\{(2a + x)/2a\}^{-1/2}$	M1 A1		
	Neglect $(x/a)^2$ and higher powers:	$d^2x/dt^2 = g(1 - x/2a) - g(1 - x/4a) = - gx/4a$	M1 A1		
	Find period from $T = 2\pi/\omega$:	$T = 2\pi\sqrt{(4a/g)}$ *or* $4\pi\sqrt{(a/g)}$	B1	5	
	Find v from $v^2 = \omega^2 (A^2 - x^2)$:	$v^2 = (g/4a)\,\{(a/20)^2 - (a/40)^2\}$	M1		
		$v = (1/80)\sqrt{(3ag)}$ **A.G.**	A1		
	Find t from $x = a \cos \omega t$ (A.E.F.):	$t = \sqrt{(4a/g)} \cos^{-1} \tfrac{1}{2} = \tfrac{2}{3}\pi\sqrt{(a/g)}$	M1 A1	4	**9**
5	Find MI of $ABCD$ about O:	$I_{ABCD} = \tfrac{1}{3}m\{(2a)^2 + a^2\} = 5ma^2/3$	B1		
	Find MI of $EFGH$ about O:	$I_{EFGH} = \tfrac{1}{3}(\tfrac{1}{4}m)\{a^2 + (\tfrac{1}{2}a)^2\} = 5ma^2/48$	M1 A1		
	EITHER: Find MI of final lamina about O:	$I_O = I_{ABCD} - I_{EFGH'} = 25ma^2/16$	M1 A1		
	Find MI of final lamina about A:	$I_A = I_O + \tfrac{3}{4} m \times 5a^2$	M1 A1		
		$= 85ma^2/16$ **A.G**	A1		
	OR: Find MI of $ABCD$ about A:	$I'_{ABCD} = I_{ABCD} + 5ma^2 = 20ma^2/3$	(M1 A1)		
	Find MI of $EFGH$ about A:	$I'_{EFGH} = I_{EFGH} + \tfrac{1}{4}m \times 5a^2 = 65ma^2/48$	(M1 A1)		
	Find MI of final lamina about A:	$I_A = I_{EFGH} - I'_{EFGH} = 85ma^2/16$ **A.G**	(B1)	8	
	State or use $u = (\sqrt{20})\, a\omega$		B1		
	Use energy when C above A to find ω_{min}:	$\tfrac{1}{2}I_A\, \omega_{min}^2 = \tfrac{3}{4}\, mg \times (\sqrt{20})\, a$	M1 A1		
	Hence find u_{min}^2:	$u_{min}^2 = 20\,(32/85)(\tfrac{3}{4}\sqrt{20})\, ag$	M1		
		$= (192\sqrt{5}/17)\, ag$ **A.G.**	A1	5	**13**
6 (i)	Sketch 6 randomly scattered points (lose B1 here if not 6 points)		B1		
(ii)	Sketch 6 (or more) points as if on negative gradient line		B1	2	
	Show on (i) line labelled y on x between points and approx. horizontal		B1		
	Show on (i) line labelled x on y between points and approx. vertical		B1	2	
	Show on (ii) line labelled y on x passing through points		B1		
	State or show on (ii) that x on y coincides with y on x		B1	2	**6**

7		Consider differences e.g. course 2 – course 1:	1·7 2·6 –1·1 –1·7 2·0 0·5	M1		
		Calculate sample mean:	$\bar{d}$ = 4 / 6 = 0·6667	M1		
		Estimate population variance: (allow biased here: 23/9 *or* 2·556 *or* $1{\cdot}599^2$)	$s^2 = (18 - 4^2/6) / 5$ = 46/15 *or* 3·067 *or* $1{\cdot}751^2$	M1		
		State hypotheses (A.E.F.), e.g.:	H_0: $\mu_1 - \mu_2 = 0$, H_1: $\mu_1 - \mu_2 \neq 0$	B1		
		Calculate value of t (to 3 s.f.):	$t = \bar{d}/(s/\sqrt{6})$ = 0·932[5]	M1 A1		
		State or use correct tabular t value: (or can compare $\bar{d}$ with 1.44)	$t_{5,\,0.95}$ = 2·01[5]	*B1		
		Correct conclusion (AEF, √ on t, dep *B1):	No difference between mean times	B1√	8	8
8		Find both sample means:	$\bar{x} = 29{\cdot}2$, $\bar{y} = 24{\cdot}4$	B1		
		Estimate both popn. variances (to 4 s.f.)	$s_x^2 = (55500 - 1752^2/60) / 59$			
			and $s_y^2 = (33500 - 1220^2/50) / 49$	M1 A1		
		(allow biased here: 72·36 *and* 74·64)	$s_x^2 = 73{\cdot}59$ *and* $s_y^2 = 76{\cdot}16$	A1		
		EITHER: Estimate combined variance (3 s.f.):	$s^2 = s_x^2/60 + s_y^2/50$ = 2·750 *or* $1{\cdot}658^2$	M1 A1		
		Use this s to find conf. interval:	$(\bar{x} - \bar{y}) \pm 1{\cdot}96\, s$	M1 A1		
		Evaluate:	4·8 ± 3·25 *or* [1·55, 8·05]	A1		
		OR: Estimate common variance (to 3 s.f.):	$s^2 = (59\,s_x^2 + 49\,s_y^2) / 108$			
		(note s_x and s_y not needed explicitly)	*or* $(55500 - 1752^2/60 +$ $33500 - 1220^2/50) / 108$			
			= 74·8 *or* $8{\cdot}65^2$ *or* 3364/45	(M1 A1)		
		Use this s to find conf. interval:	$(\bar{x} - \bar{y}) \pm 1{\cdot}96\, s\, \sqrt{(1/60 + 1/50)}$	(M1 A1)		
		Evaluate:	4·8 ± 3·24[5] *or* [1·55[5], 8·04[5]]	(A1)	9	9
9	(i)	Find n using gradient:	$-\frac{3}{4} = (192 - 24 \times 34/n) / (160 - 24^2/n)$ $4992/n = 1248$, $n = 4$	M1 A1 A1	3	
	(ii)	Find regression line $x = Ay + B$: M0 for $y - 34/4 = -\frac{3}{4}(x - 24/4)$	$x - 24/4 = (-12/35)(y - 34/4)$ $x = -12y/35 + 312/35$ *or* $-0{\cdot}343y + 8{\cdot}91$	M1 A1	2	
	(iii)	Find coefficient r (ignore sign for M1): State regression line (B1 for each term on RHS):	$r = -12/\sqrt{(16 \times 35)}$ *or* $-\sqrt{(\frac{3}{4} \times 12/35)}$ $= -0{\cdot}507$ *or* $-3\sqrt{35}/35$ *or* $-3/\sqrt{35}$	 M1 A1	2	9

10 **(i)**	Find prob. of Jill scoring on her 5th kick:	$(⅔)^4\,(¾)^4\,⅓ = 1/48$ A.G	M1 A1	2	
(ii)	Find prob. of Kate winning on her nth kick:	$(⅔)^{n-1}\,(¾)^{n-1}\,⅔\,¼$	M1 A1		
		$= 1/(3 \times 2^n)$ **A.G.**	A1	3	
(iii)	Find prob. of Jill winning game:	$⅓\,¾\,\{1 + (⅔\,¾) + (⅔\,¾)^2 + \ldots\}$	M1 A1		
		$= ¼\,(1 + ½ + ¼ + \ldots) = ½$	A1	3	
(iv)	Find prob that game is a draw	$⅓\,¼\,\{1 + (⅔\,¾) + (⅔\,¾)^2 + \ldots\}$	M1 A1		
		$= ⅓\,¼\,(1 + ½ + ¼ + \ldots)$			
		or 1 – P(Jill or Kate wins)			
		$= 1 - \{½ + ⅓(½ + ¼ + \ldots)\}$	(M1 A1)		
		$= 1/6$	A1	3	**11**
11 **(a)**	Resolve horizontally and vertically:	$F_A = R_B$	B1		
	Relate limiting frictions and reactions at A and B:	$F_A = 2\mu R_A$ *and* $F_B = \mu R_B$	B1		
	EITHER: Resolve vertically :	$F_B + R_A = W$	B1		
	Combine eqns to find R_A:	$R_A = W / (1 + 2\mu^2)$	M1 A1		
	Take moments about B	$2R_A \cos\theta - 2F_A \sin\theta = W\cos\theta$	M1 A1		
	Rearrange and use $F_A = 2\mu R_A$	$R_A - 2\mu R_A \tan\theta = ½W$			
		$R_A = ½W / (1 - 2\mu \tan\theta)$	A1		
	Combine to find $\tan\theta$:	$\tan\theta = (1 - 2\mu^2)/4\mu$ **A.G**	M1 A1		
	OR: Resolve vertically:	$F_B + R_A = W$	(B1)		
	Combine eqns to find R_B:	$R_B = 2\mu W / (1 + 2\mu^2)$	(M1 A1)		
	Take moments about A:	$2R_B \sin\theta + 2F_B \cos\theta = W\cos\theta$	(M1 A1)		
	Rearrange and use $F_B = \mu R_B$	$R_B \tan\theta + \mu R_B = ½W$			
		$R_B = ½W / (\tan\theta + \mu)$	(A1)		
	Combine to find $\tan\theta$:	$\tan\theta = (1 - 2\mu^2)/4\mu$ **A.G.**	(M1 A1)		
	OR: Take moments about centre of rod:	$(F_A + R_B)\sin\theta = (R_A - F_B)\cos\theta$	(M2 A1 A1)		
	Use first 3 eqns to eliminate 3 forces, e.g.:	$\tan\theta = (R_B/2\mu - \mu R_B)/(R_B + R_B)$	(M1 A1 A1)		
		$= (1 - 2\mu^2)/4\mu$ **A.G.**	(A1)	10	

	Find positive value of μ when $\theta = 45°$:	$2\mu^2 + 4\mu - 1 = 0$ $\mu = \sqrt{(3/2)} - 1$ *or* $0{\cdot}225$	A.E.F M1 A1		
	Find positive value of μ when $\theta = 0°$:	$\mu = 1/\sqrt{2}$ *or* $0{\cdot}707$	A.E.F B1		
	State set of possible values of μ (A.E.F.):	$[\sqrt{(3/2)} - 1, 1/\sqrt{2}]$ *or* $[0{\cdot}225, 0{\cdot}707]$	A1	4	**14**
(b)	Tabulate observed values:	*Rural* *Urban* *Support* A $60 - A$ *Not support* $45 - A$ $A - 5$	M1 A1		
	Find corresponding expected values:	27 33 18 22	M1 A1		
	Calculate value of χ^2:]	$\chi^2 = (27 - A)^2/27 + (A - 27)^2/18$ $+ (A - 27)^2/33 + (27 - A)^2/22$	M1		
		$= (50/297)(A - 27)^2$ *or* $0{\cdot}168[35]\,(A - 27)^2$	A1		
	State or use correct tabular χ^2 value:	$\chi_{tab}{}^2 = \chi_{1,\,0.9}{}^2 = 2{\cdot}706$ (to 3 s.f.)	B1		
	Use conclusion of independence to find eqn. for A:	$(50/297)(A - 27)^2 < \chi_{tab}{}^2$	M1		
	Find bounds for A (to 3 s.f.): (integer valued reqd for this A1)	$(A - 27)^2 < 16{\cdot}07$ $A_{min} = 23$ *and* $A_{max} = 31$	A1 A1	10	
	Relate new value $\chi_{new}{}^2$ to original χ^2:	$\chi_{new}{}^2 = N \times \chi^2$	M1		
	Use conclusion of independence to find eqn. for A:	$0{\cdot}168[35]\,N(A - 27)^2 < \chi_{tab}{}^2$	M1		
	Find N_{max} with $A = 29$: (integer value reqd for this A1)	$N < 2{\cdot}706/(4 \times 0{\cdot}16835) = 4{\cdot}02$ $N_{max} = 4$	A1 A1	4	**14**